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Technical Notes

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Supersonic Indicial Lift Functions from Transform Methods

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Nomenclature

a = speed of sound
C_L, $\bar{C}_L$ = lift coefficient, lift history due to $f(t)$
c = chord length
$\mathcal{F}[\cdot]$ = Fourier transform of $[\cdot]$
$f(t)$ = nondimensional downwash time history
$\mathcal{H}(t)$ = Heaviside function
i = complex constant, $\sqrt{-1}$
$J_n[\cdot]$ = Bessel function of the first kind, order n
L = lift on the airfoil in the positive z direction
$\mathcal{L}[\cdot]$ = Laplace transform of $[\cdot]$
M = freestream Mach number, U_∞/a_∞
p = linear pressure perturbation
s = Laplace transform variable
T = convolution variable, $t-\tau$
$\tilde{t}$, t = dimensional time, scaled by c/a_∞
U = streamwise fluid velocity
$\tilde{W}$ = Laplace transform of downwash, $\mathcal{L}[\tilde{w}]$
$\tilde{w}$, w = dimensional airfoil downwash distribution in the negative z direction, scaled by U_∞
$\tilde{x}$, x = dimensional streamwise coordinate, scaled by c
$\tilde{z}$ = dimensional vertical coordinate
θ, ψ = variables to change x and T integrations, respectively
μ = transform expression
ρ = fluid density
$\tilde{\sigma}$, σ = Fourier transform variable, scaled by $1/c$
$\tilde{\tau}$, τ = dimensional dummy time variable, scaled by c/a_∞
ϕ, Φ = velocity potential function, $\mathcal{L}[\phi]$
∇^2 = Laplacian operator

Subscripts

I, II, III = time regime for C_L
∞ = freestream condition

Superscripts

(n) = order of downwash polynomial used to compute C_L
$*$ = Fourier transformed variable
-1 = inverse operation

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I. Introduction

INDICIAL lift is a time-dependent aerodynamic response to a step change in airfoil motion. For sufficiently small motions, the indicial lift function is a linear perturbation of the airfoil downwash distribution from an established steady state. In the special cases of subsonic and supersonic flow, the linearity of the governing equations allows the lift due to arbitrary motions of sufficiently thin airfoils to be exactly constructed by superposition. Aerodynamic load calculation in the inherently nonlinear transonic regime also benefits from indicial function approximations, because the solution of the flowfield is calculated only once instead of at each parameter value (e.g., frequency) of interest [1]. It follows that indicial methods offer a means to reduce the cost of computational fluid dynamic calculations [2,3].

Lomax et al. [4] first determined the linear indicial functions due to pitch and plunge motions, for which the supersonic case was verified by Bisplinghoff et al. [5] using the source-pulse method. An alternative derivation of the indicial functions using Laplace transforms was considered by Bisplinghoff et al. and later extended by Edwards et al. [6] and Dowell et al. [7] for arbitrary airfoil motions; the implications of these latter developments for analytic solutions are considered in this work.

This Note presents an alternative method to evaluate the classical linear indicial lift of airfoils in supersonic flow. For an arbitrary airfoil downwash distribution, the present formulation reduces the calculation of transient lift to quadrature. It is also demonstrated that analytic solutions exist for polynomial spatial distributions of downwash. The transition from "piston theory" at short times to steady-state solutions is carried out analytically for zeroth- to cubic-order polynomial downwash functions and is validated by available results in the literature for constant and linear downwash spatial distributions. New results for the aerodynamic loads of simply supported panel-flutter modes are also presented and compared with analytic polynomial approximations.

II. Analysis

The lift of a thin airfoil in supersonic potential flow can be found by solving the convective wave equation with a downwash condition on the airfoil. Note that the sign convention on the downwash is positive down.

$$\nabla^2_{\tilde{x},\tilde{z}}\phi - \frac{1}{a_\infty^2}\left(\frac{\partial}{\partial\tilde{t}} + U_\infty\frac{\partial}{\partial\tilde{x}}\right)^2\phi = 0 \tag{1}$$

$$\left.\frac{\partial\phi}{\partial\tilde{z}}\right|_{\tilde{z}=0} = -\,\tilde{w}(\tilde{x}) \tag{2}$$

Apply Laplace and Fourier transforms to both Eqs. (1) and (2). The transform pairs are defined as

$$\mathcal{L}[\phi(\tilde{x},\tilde{z},\tilde{t})]=\int_0^\infty \phi e^{-s\tilde{t}}\,\mathrm{d}\tilde{t} \Leftrightarrow \mathcal{L}^{-1}[\Phi(\tilde{x},\tilde{z},s)]=\int_0^\infty \Phi e^{s\tilde{t}}\,\mathrm{d}s \tag{3}$$

$$\mathcal{F}[\phi(\tilde{x},\tilde{z},\tilde{t})]=\int_{-\infty}^\infty \phi e^{-i\tilde{\sigma}\tilde{x}}\,\mathrm{d}\tilde{x} \Leftrightarrow \mathcal{F}^{-1}[\phi^*(\tilde{\sigma},\tilde{z},\tilde{t})]=\int_{-\infty}^\infty \phi^* e^{i\tilde{\sigma}\tilde{x}}\,\mathrm{d}\tilde{\sigma} \tag{4}$$

Therefore, the transformed expressions of Eqs. (1) and (2) are

$$\frac{\mathrm{d}^2\Phi^*}{\mathrm{d}\tilde{z}^2}-\mu^2\Phi^*=0 \tag{5}$$

$$\left.\frac{\mathrm{d}\Phi^*}{\mathrm{d}\tilde{z}}\right|_{\tilde{z}=0}=-\tilde{W}^* \tag{6}$$

where

$$\mu^2=-(M^2-1)\tilde{\sigma}^2+\frac{2M}{a_\infty}(i\tilde{\sigma})s+\frac{s^2}{a_\infty^2} \tag{7}$$

As outlined in Bisplinghoff et al. [5], the antisymmetry of the problem may be exploited by considering only points above the airfoil (i.e., $\tilde{z}\geq 0^+$). Apply boundary condition Eq. (6) to Eq. (5) and assert boundedness as $\tilde{z}\to\infty$ to show that the potential function evaluated at the airfoil surface is proportional to the downwash in transform space.

$$\Phi^*|_{\tilde{z}=0}=\frac{\tilde{W}^*}{\mu} \tag{8}$$

Invert the Laplace transform and apply the convolution theorem

$$\mathcal{L}^{-1}[F(s)\cdot G(s)]=\int_0^{\tilde{t}} f(\tilde{t}-\tilde{\tau})g(\tilde{\tau})\,\mathrm{d}\tilde{\tau} \tag{9}$$

to arrive at an integral form of the potential function in terms of the downwash distribution.

$$\phi^*|_{\tilde{z}=0^+}=a_\infty\int_0^{\tilde{t}}\tilde{w}^*(\tilde{\tau})e^{-i\tilde{\sigma}Ma_\infty(\tilde{t}-\tilde{\tau})}J_0[a_\infty\tilde{\sigma}(\tilde{t}-\tilde{\tau})]\,\mathrm{d}\tilde{\tau} \tag{10}$$

The linear pressure perturbation from Bernoulli's equation

$$p=-\rho_\infty\left(\frac{\partial\phi}{\partial\tilde{t}}+U_\infty\frac{\partial\phi}{\partial\tilde{x}}\right) \tag{11}$$

relates the potential function and downwash distribution to the airfoil surface by [7]

$$p^*=-\rho_\infty\left(\frac{\partial\phi^*}{\partial\tilde{t}}+(i\tilde{\sigma})U_\infty\phi^*\right) \tag{12}$$

$$=-\rho_\infty a_\infty\tilde{w}^*(\tilde{\sigma},\tilde{t})+\rho_\infty a_\infty^2\int_0^{\tilde{t}}\tilde{\sigma}\tilde{w}^*(\tilde{\sigma},\tilde{\tau})e^{-i\tilde{\sigma}Ma_\infty(\tilde{t}-\tilde{\tau})} \times J_1[a_\infty\tilde{\sigma}(\tilde{t}-\tilde{\tau})]\,\mathrm{d}\tilde{\tau} \tag{13}$$

$$\equiv p_0^*+p_1^* \tag{14}$$

The total lift is found by integrating the pressure field acting on the airfoil.

$$L\equiv -2\int_0^c p\,\mathrm{d}\tilde{x} \tag{15}$$

$$=2\rho_\infty a_\infty\int_0^c\tilde{w}\,\mathrm{d}\tilde{x}-\frac{1}{\pi}\int_{-\infty}^\infty p_1^*\left(\frac{e^{i\tilde{\sigma}c}-1}{i\tilde{\sigma}}\right)\mathrm{d}\tilde{\sigma} \tag{16}$$

Substitute the expression for p_1^* from Eq. (13) into Eq. (16) to complete the general formulation of the lift of a supersonic, small-disturbance airfoil with an arbitrary downwash as a function of both time and the streamwise coordinate $\tilde{x}$.

$$L=2\rho_\infty a_\infty\int_0^c\tilde{w}(\tilde{x},\tilde{t})\,\mathrm{d}\tilde{x} +\frac{i\rho_\infty a_\infty^2}{\pi}\int_0^{\tilde{t}}\int_0^c\int_{-\infty}^\infty\tilde{w}(\tilde{x},\tilde{\tau})e^{-i\tilde{\sigma}[Ma_\infty(\tilde{t}-\tilde{\tau})+\tilde{x}-c]}(e^{i\tilde{\sigma}c}-1) \times J_1[a_\infty\tilde{\sigma}(\tilde{t}-\tilde{\tau})]\,\mathrm{d}\tilde{\sigma}\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{\tau} \tag{17}$$

Nondimensionalize Eq. (17) using the chord length and the ratio of the chord to the freestream speed of sound, c/a_∞, as the respective length and time scales. Also change the time integration variable to T and scale the downwash velocity by U_∞. The section lift coefficient

$$C_L\equiv\frac{L}{\frac{1}{2}\rho_\infty U_\infty^2 c} \tag{18}$$

follows from Eq. (17).

$$C_L=\frac{4}{M}\int_0^1 w(x,t)\,\mathrm{d}x+\frac{2i}{\pi M}\int_0^t\int_0^1\int_{-\infty}^\infty w(x,T)(e^{-i\sigma(MT+x-1)} - e^{-i\sigma(MT+x)})J_1[\sigma T]\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}T \tag{19}$$

The classical piston theory result of $4/M$ is observed in Eq. (19) for $t=0$.

It is clear that the principal difficulty in calculating the lift coefficient is evaluating the so-called memory effect. The integration of this expression with respect to the transform variable exists in closed form for $M>1$. Note the elimination of odd integrands.

$$\int_0^\infty\sin[\sigma(MT+x)]J_1[\sigma T]\,\mathrm{d}\sigma=0\quad\text{for } M>1 \tag{20}$$

$$\int_0^\infty\sin[\sigma(MT+x-1)]J_1[\sigma T]\,\mathrm{d}\sigma =\begin{cases}\frac{MT+x-1}{T\sqrt{T^2-(MT+x-1)^2}} & T^2>(MT+x-1)^2\\ 0 & \text{otherwise}\end{cases} \tag{21}$$

The region for nontrivial solutions, shown in Fig. 1, is the intersection of $x\in[0,1]$ and $T^2>(MT+x-1)^2$. The times $t=1/(M+1), 1/(M-1)$ correspond to the events at which signals from the leading edge reach the trailing edge of the airfoil. It is evident that two transient regimes (I and II) exist within $0\leq t\leq 1/(M-1)$ and there is a steady-state range (III) for $t>1/(M-1)$.

Herein, only indicial lift cases are considered, that is,

$$w(x,t)=w(x)\mathcal{H}(t) \tag{22}$$

The downwash is impulsively changed at $t=0$ and remains time-invariant for $t>0$. By assuming the downwash history as Eq. (22), the lift coefficient from Eq. (19) becomes the nondimensional indicial lift function. In general, arbitrary time-dependent solutions

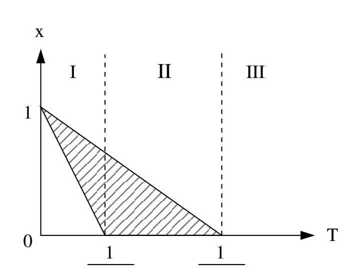

Fig. 1 Memory-effect integration region.

may be constructed using these indicial functions in conjunction with Duhamel's integral [8]. If the Heaviside function of Eq. (22) is replaced by an arbitrary velocity schedule $f(t)$ for $t \geq 0$, the lift history $\bar{C}_L(t)$ is the following.

$$\bar{C}_L(t) = f(0)C_L(t) + \int_0^t \frac{\mathrm{d}f}{\mathrm{d}t}(\tau)C_L(t-\tau)\,\mathrm{d}\tau \tag{23}$$

$$= f(t)C_L(0) + \int_0^t f(\tau)\frac{\mathrm{d}C_L}{\mathrm{d}t}(t-\tau)\,\mathrm{d}\tau \tag{24}$$

Equation (24) holds for discontinuous $f(t)$, whereas Eq. (23) will not.

A. Transient Lift

Consider the first transient regime where $0 \leq t \leq 1/(M+1)$. The integration bounds for x are

$$1-(M+1)T < x < 1-(M-1)T \tag{25}$$

Let

$$\sin\theta \equiv M - \frac{1-x}{T} \tag{26}$$

and change the integration of x to an integration with respect to θ. The indicial lift expression simplifies to

$$(C_L)_{\mathrm{I}} = \frac{4}{M}\int_0^1 w(x)\,\mathrm{d}x + \frac{4}{\pi M}\int_0^t \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} w(\theta,T)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}T \tag{27}$$

For the second transient regime, the integration is performed across a choice of two regions, as shown in Fig. 2. Integrating over regions A′ and B′ implies a change of the integration in x, whereas regions A and B denote a change of the integration in T. The bounds for each region are

Region A:

$$\begin{cases} T \in \left[\frac{1-x}{M-1}, t\right] \\ x \in [0, 1-(M-1)t] \end{cases} \tag{28}$$

Region B:

$$\begin{cases} T \in \left[\frac{1-x}{M+1}, \frac{1-x}{M-1}\right] \\ x \in [1-(M-1)t, 1] \end{cases} \tag{29}$$

Region A′:

$$\begin{cases} x \in [1-(M+1)T, 1-(M-1)T] \\ T \in \left[0, \frac{1}{M+1}\right] \end{cases} \tag{30}$$

Region B′:

$$\begin{cases} x \in [0, 1-(M-1)T] \\ T \in \left[\frac{1}{M+1}, t\right] \end{cases} \tag{31}$$

Change the integration of T to ψ by defining

$$\sin\psi \equiv M - \frac{1-x}{T} \tag{32}$$

Although this replacement is rather tedious for the general transient case, it will later be shown that this change of variables leads to an analytic form of the steady-state indicial lift for all downwash distributions. Alternatively, the integration of x may be replaced as in the analysis of the first transient regime. The indicial lift function is presented for each integration scheme.

$$(C_L)_{\mathrm{II}} = \frac{4}{M}\int_0^1 w(x)\,\mathrm{d}x + \frac{4}{\pi M}\left\{\int_0^{\frac{1}{M+1}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} w(\theta,T)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}T + \int_{\frac{1}{M+1}}^{t}\int_{\arcsin[M-\frac{1}{T}]}^{\frac{\pi}{2}} w(\theta,T)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}T\right\} \tag{33}$$

$$= \frac{4}{M}\int_0^1 w(x)\,\mathrm{d}x + \frac{4}{\pi M}\left\{\int_0^{1-(M-1)t}\int_{-\frac{\pi}{2}}^{\arcsin[M-\frac{1-x}{t}]} w(x)\frac{\sin\psi}{M-\sin\psi}\,\mathrm{d}\psi\,\mathrm{d}x + \int_{1-(M-1)t}^{1}\int_{\arcsin[M-\frac{1}{t}]}^{\frac{\pi}{2}} w(x)\frac{\sin\psi}{M-\sin\psi}\,\mathrm{d}\psi\,\mathrm{d}x\right\} \tag{34}$$

B. Steady State

For $t > 1/(M-1)$, the indicial lift achieves a steady-state value. It is easily seen from Eq. (34) that as $t \to 1/(M-1)$, the indicial function is expressed simply as

$$(C_L)_{\mathrm{III}} = \frac{4}{M}\int_0^1 w(x)\,\mathrm{d}x + \frac{4}{\pi M}\int_0^1\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} w(x)\frac{\sin\psi}{M-\sin\psi}\,\mathrm{d}\psi\,\mathrm{d}x \tag{35}$$

Here, the downwash is strictly a function of the spanwise coordinate, which allows the memory effect to be found in closed form.

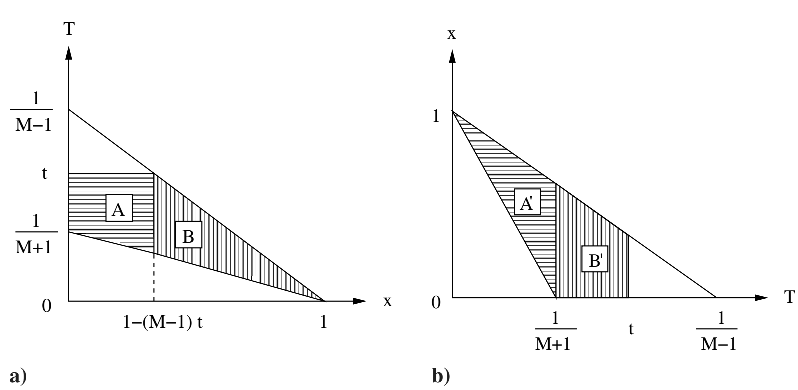

Fig. 2 Integration regions for the second transient regime, $t \in [1/(M+1), 1/(M-1)]$.

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin\psi}{M-\sin\psi}\,\mathrm{d}\psi = \pi\left(\frac{M}{\sqrt{M^2-1}}-1\right) \tag{36}$$

$$(C_L)_{\mathrm{III}} = \frac{4}{\sqrt{M^2-1}}\int_0^1 w(x)\,\mathrm{d}x \tag{37}$$

III. Application

A. Polynomial Downwash

In this section, the general indicial lift expressions (27), (33), (34), and (37), are further developed analytically for a polynomial downwash distribution $w(x) = x^n$. The lift in the first transient region is readily produced by substituting $x^n = [1 - T(M - \sin\theta)]^n$ into Eq. (27) and integrating first by T.

$$(C_L)_{\mathrm{I}}^{(n)} = \frac{4}{(n+1)M}\left[\frac{M}{\sqrt{M^2-1}} - \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sin\theta}{M-\sin\theta} \times [1 - t(M-\sin\theta)]^{n+1}\,\mathrm{d}\theta\right] \tag{38}$$

The second transient regime is best integrated first in the dummy variable ψ after changing from the time integration. After some algebraic manipulation,

$$(C_L)_{\mathrm{II}}^{(n)} = \frac{4}{(n+1)M}\left\{\frac{M}{\sqrt{M^2-1}} - [1-(M-1)t]^{n+1} \times \left[\frac{M}{\sqrt{M^2-1}}\left(1 - \frac{2}{\pi}\arctan\sqrt{\frac{M+1}{M-1}}\right) - \frac{1}{2}\right] - \frac{n+1}{\pi}\int_0^{1-(M-1)t} x^n \times \left(\frac{2M}{\sqrt{M^2-1}}\arctan\left[\frac{1 - M\tan[\frac{1}{2}\arcsin(M-[(1-x)/t])]}{\sqrt{M^2-1}}\right] + \arcsin\left[M - \frac{1-x}{t}\right]\right)\mathrm{d}x\right\} \tag{39}$$

By observation, the limit of Eq. (39) as $t \to 1/(M-1)$ is

$$(C_L)_{\mathrm{III}}^{(n)} = \frac{4}{(n+1)\sqrt{M^2-1}} \tag{40}$$

The analytic expressions of C_L for polynomial downwash distributions of zeroth to cubic orders are found in the Appendix. Figure 3 displays these expressions for $M = 1.2$. Also, the indicial lift curve for $n = 10$ is plotted using numerical quadrature.

B. Panel-Flutter Modes

This section develops the general indicial aerodynamic loads for a sinusoidal downwash distribution, $w(x) = \sin(n\pi x)$, representative of a simply supported two-dimensional panel-flutter mode. Following the same procedure as with the polynomial downwash, the indicial lift functions within the three regions are

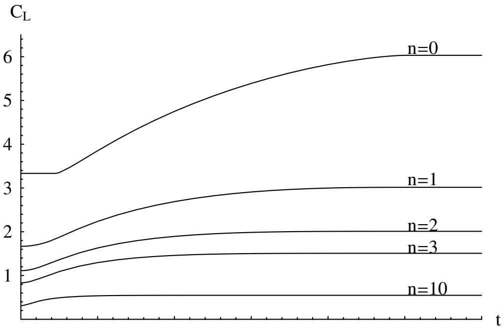

Fig. 3 Indicial lift curves of $w(x) = x^n$ for $M = 1.2$.

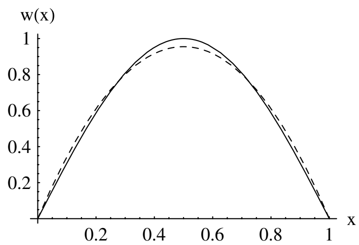

Fig. 4 Downwash distribution for the first panel mode; exact (solid line) and polynomial (dashed line).

$$(C_L)_{\mathrm{I}} = \frac{4}{n\pi M}\left[1 + (-1)^n\left\{\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sin\xi}{M-\sin\xi}\cos[n\pi t(M-\sin\xi)]\,\mathrm{d}\xi - \frac{M}{\sqrt{M^2-1}}\right\}\right] \tag{41}$$

$$(C_L)_{\mathrm{II}} = \frac{4}{n\pi M}\left(\frac{1}{2}\{1-(-1)^n\cos[n\pi(M-1)t]\} - \frac{(-1)^n M}{\sqrt{M^2-1}}\left(1 - \cos[n\pi(M-1)t] - \frac{2}{\pi}\{(-1)^n - \cos[n\pi(M-1)t]\}\arctan\sqrt{\frac{M+1}{M-1}}\right) - n\int_0^{1-(M-1)t} \times \sin(n\pi x)\left(\frac{2M}{\sqrt{M^2-1}}\arctan\left[\frac{1 - M\tan[\frac{1}{2}\arcsin(M-[(1-x)/t])]}{\sqrt{M^2-1}}\right] + \arcsin\left[M - \frac{1-x}{t}\right]\right)\mathrm{d}x\right) \tag{42}$$

$$(C_L)_{\mathrm{III}} = \frac{4}{n\pi\sqrt{M^2-1}}[1-(-1)^n] \tag{43}$$

Using the closed-form polynomial solutions tabulated in the Appendix, it is possible to construct analytic approximations to the panel-mode solutions by linear superposition. Illustrative examples of these approximate solutions are made here for the first and second panel modes.

The first panel mode may be approximated as a quadratic downwash description, and the question arises as to what is the proper scaling factor to match the lift coefficients. This is easily done by matching the steady-state lift values, which have simple analytic expressions for both the polynomial approximation and the exact panel-mode shape. Thus, the following approximation is made in which the maximum value over $x \in [0, 1]$ is $3/\pi$.

$$\sin(\pi x) \approx \frac{12}{\pi}(x - x^2) \tag{44}$$

Apply the same maximum value to the cubic approximation of the second panel mode.

$$\sin(2\pi x) \approx \frac{18\sqrt{3}}{\pi}(x - 3x^2 + 2x^3) \tag{45}$$

The first and second panel-mode shapes are plotted in Figs. 4 and 5, respectively, with their approximations from Eqs. (44) and (45). Representative comparisons of the approximate indicial lift curves with exact results are made in Figs. 6 and 7 for $M = 1.2$, 1.4, and 2.0.

IV. Conclusions

A general framework is developed to calculate the nondimensional indicial lift due to the presence of an arbitrary airfoil downwash distribution in supersonic flow. The indicial lift functions are derived and reduced to quadrature. Special attention is given to

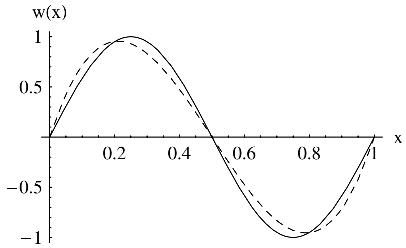

Fig. 5 Downwash distribution for the second panel mode; exact (solid line) and polynomial (dashed line).

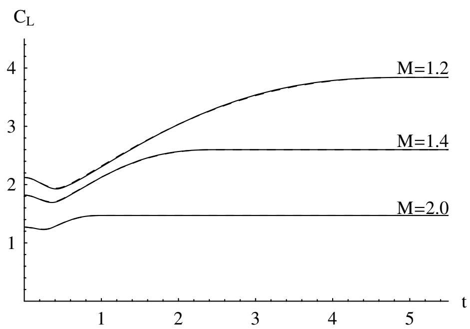

Fig. 6 Indicial lift of first panel mode; exact (solid line) and polynomial (dashed line).

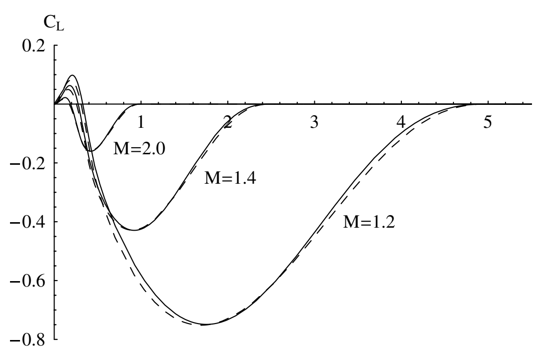

Fig. 7 Indicial lift of second panel mode; exact (solid line) and polynomial (dashed line).

sinusoidal and polynomial downwash distributions. Analytic results for zeroth- through cubic-order polynomial downwash expressions are tabulated in the Appendix, and their utility to approximate more sophisticated downwash distributions is demonstrated for the first two simply supported panel-mode shapes.

Appendix: Indicial Functions for Polynomial Downwash

The indicial lift functions for polynomial downwash distributions $w(x) = x^n$ are herein denoted as $(C_L)^{(n)}_{(\mathrm{I,II,III})}$. The superscript denotes the polynomial order, and the subscript is the time region for which the solution is applicable.

The following results for $n = 0{,}1$ are verified by comparison to those of Lomax et al. [4] and Bisplinghoff et al. [5].

$$(C_L)^{(0)}_{(\mathrm{I})} = \frac{4}{M} \tag{A1}$$

$$(C_L)^{(0)}_{(\mathrm{II})} = \frac{4}{\pi M}\left\{\arccos\left[M - \frac{1}{t}\right] + \frac{M}{\sqrt{M^2-1}}\arccos[M - (M^2-1)t] + t\sqrt{1 - \left(M - \frac{1}{t}\right)^2}\right\} \tag{A2}$$

$$(C_L)^{(0)}_{(\mathrm{III})} = \frac{4}{\sqrt{M^2-1}} \tag{A3}$$

$$(C_L)^{(1)}_{(\mathrm{I})} = \frac{2}{M}\left(1 + \frac{t^2}{2}\right) \tag{A4}$$

$$(C_L)^{(1)}_{(\mathrm{II})} = \frac{t^2+2}{\pi M}\arccos\left[M - \frac{1}{t}\right] + \frac{2}{\pi\sqrt{M^2-1}}\arccos[M - (M^2-1)t] - \frac{t}{\pi M}(Mt-3)\sqrt{1 - \left(M - \frac{1}{t}\right)^2} \tag{A5}$$

$$(C_L)^{(1)}_{(\mathrm{III})} = \frac{2}{\sqrt{M^2-1}} \tag{A6}$$

The new results for $n = 2$, 3 come from direct integration of Eqs. (33), (38), and (40).

$$(C_L)^{(2)}_{(\mathrm{I})} = \frac{4}{3M}\left(1 + \frac{3t^2}{2} - Mt^3\right) \tag{A7}$$

$$(C_L)^{(2)}_{(\mathrm{II})} = \frac{4 - 2t^2(2Mt-3)}{3\pi M}\arccos\left[M - \frac{1}{t}\right] + \frac{4}{3\pi\sqrt{M^2-1}}\arccos[M - (M^2-1)t] + \frac{2t}{9\pi M}[2(Mt-1)^2 - 3(Mt-1) + 2(2t^2+3)]\sqrt{1 - \left(M - \frac{1}{t}\right)^2} \tag{A8}$$

$$(C_L)^{(2)}_{(\mathrm{III})} = \frac{4}{3\sqrt{M^2-1}} \tag{A9}$$

$$(C_L)^{(3)}_{(\mathrm{I})} = \frac{1}{M}\left\{1 + \frac{t^2}{8}[12(Mt-1)^2 - 8(Mt-1) + 3t^2 + 4]\right\} \tag{A10}$$

$$(C_L)^{(3)}_{(\mathrm{II})} = \frac{8 + t^2[12(Mt-1)^2 - 8(Mt-1) + 3t^2 + 4]}{8\pi M} \times \arccos\left[M - \frac{1}{t}\right] + \frac{t}{24\pi M}\sqrt{1 - \left(M - \frac{1}{t}\right)^2}[6(Mt-1)^3 - 8(Mt-1)^2 + (39t^2+12)(Mt-1) - 8(2t^2+3)] + \frac{1}{\pi\sqrt{M^2-1}}\arccos[M - (M^2-1)t] \tag{A11}$$

$$(C_L)^{(3)}_{(\mathrm{III})} = \frac{1}{\sqrt{M^2-1}} \tag{A12}$$

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